Absorbing Boundary Conditions and Perfectly Matched Layers for Acoustics

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Overview

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PDE Formulation

The general problem can be formulated as

$$\frac{\partial^2 u}{\partial t^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial x^2} = f(x,t), \quad x \in \Omega,\ t \in [0, T],$$
$$u(x,0) = g(x)$$

Note: NO boundary conditions!

PDE Formulation - Problems!

$$\frac{\partial^2 u}{\partial t^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial x^2} = f(x,t), \quad x \in \Omega,\ t \in [0, T],$$
$$u(x,0) = g(x)$$

- Physical Reality vs. Mathematics
  - Well-posed?
  - Does solution make sense?
- Computational Simulations
  - Computation time

Can We Force Boundary Conditions?

To make the problem well-posed we need to truncate Ω and add boundary conditions!

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial x^2} &= f(x,t), \quad x \in \Omega,\ t \in [0, T], \\ u(x,0) &= g(x) \\ u(x,t) &= ?, \quad x \in \partial\Omega \\ \frac{\partial}{\partial n} u(x,t) &= ?, \quad x \in \partial\Omega \end{aligned}$$

Dirichlet or Neumann BC?

Dirichlet

$$u(x,t) = 0, \quad x \in \partial\Omega$$

Neumann

$$\frac{\partial}{\partial n} u(x,t) = 0, \quad x \in \partial\Omega$$

But Then What?!

Absorbing Boundary condition

- Truncate domain and add BC

Perfectly Matched Layer

- Analytical continuation of the solution

Acoustics

- What is sound?
  - Compression and expansion of a fluid
- How do we express sound propagation mathematically?
  - Linear: The Linear Wave Equation
  - Non-Linear: Navier-Stokes Equation (and variations thereof)

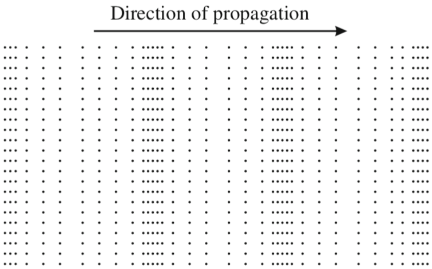


Figure 1: [?]

Linear vs Non-Linear

Linear Wave Equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial x^2} = 0,$$

Westervelt Equation

$$c^{-2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \beta\frac{\partial^3 u}{\partial t \partial x^2} = \gamma\left(\frac{\partial^2 u}{\partial t^2}\right)^2$$

Linear vs Non-Linear

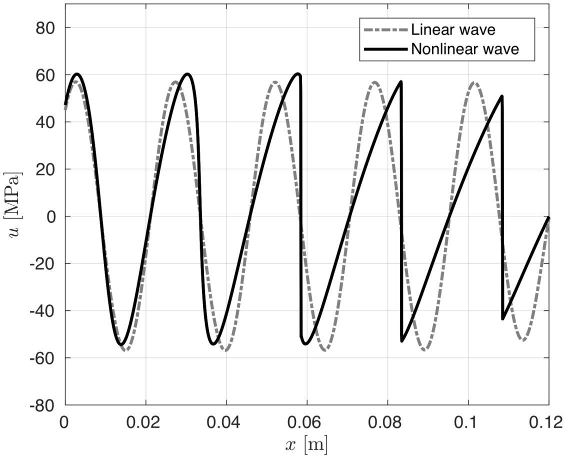


Figure 2: [?]

Absorbing Boundary Conditions

Absorbing boundary conditions (ABC) works by truncating the infinite domain to a **suitable** size and impose **artificial** boundary conditions that let waves **pass through**.

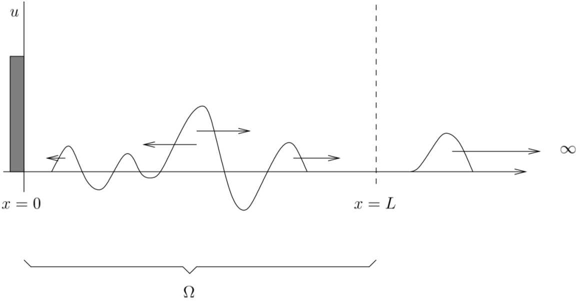


Figure 3: [?]

The Wave Equation

We consider the one-dimensionless case.

$$\frac{\partial^2 u}{\partial t^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial x^2} = 0, \quad x > 0,\ t > 0 \tag{1a}$$

$$u(x,0) = U(x) \tag{1b}$$

$$\frac{\partial}{\partial t}u(x,0) = V(x) \tag{1c}$$

$$u(0,t) = 0, \tag{1d}$$

This formulation leads to solutions with the following properties:

- Reflects at the left boundary, $x = 0$
- Propagates to the right towards infinity

But...

We need to restrict the domain to the right by adding a boundary. Hence, we need another boundary condition:

$$B(u) = b, \quad x = L,\ t > 0$$

The goal is now to find an operator B and a function b such that:

- Waves don't reflect
- The solution in the desired domain is unaffected
- The problem stays well-posed
- We can implement it on a computer (it has to be computationally efficient)

The Approach

We define two new functions

$$v = \frac{\partial u}{\partial t} + \frac{1}{c}\frac{\partial u}{\partial x}, \quad w = \frac{\partial u}{\partial t} - \frac{1}{c}\frac{\partial u}{\partial x} \tag{2}$$

These two functions satisfy

$$\begin{aligned}
\frac{\partial v}{\partial t} - \frac{1}{c}\frac{\partial v}{\partial x} &= \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t} + \frac{1}{c}\frac{\partial u}{\partial x}\right) - \frac{1}{c}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} + \frac{1}{c}\frac{\partial u}{\partial x}\right) \\
&= \frac{\partial^2 u}{\partial t^2} + \frac{1}{c}\frac{\partial^2 u}{\partial t \partial x} - \frac{1}{c}\frac{\partial^2 u}{\partial x \partial t} - \frac{1}{c^2}\frac{\partial^2 u}{\partial x^2} \\
&= \frac{\partial^2 u}{\partial t^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial x^2} \quad \longleftarrow \quad \text{Original PDE} \\
&= 0
\end{aligned}$$

And similarly

$$\frac{\partial w}{\partial t} + \frac{1}{c}\frac{\partial w}{\partial x} = 0$$

$$\frac{\partial v}{\partial t} - \frac{1}{c}\frac{\partial v}{\partial x} = 0, \quad \frac{\partial w}{\partial t} + \frac{1}{c}\frac{\partial w}{\partial x} = 0$$

Thus, we can rewrite (1a) as a system

$$\frac{\partial}{\partial t}\begin{pmatrix} v \\ w \end{pmatrix} + \frac{1}{c}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix} v \\ w \end{pmatrix} = 0. \tag{3}$$

(3) has the general solution

$$v(x,t) = \phi\left(x + \frac{1}{c}t\right), \quad w(x,t) = \psi\left(x - \frac{1}{c}t\right), \tag{4}$$

for some functions ϕ and ψ depending on the initial condition.

$$v(x,t) = \phi(x+ct), \quad w(x,t) = \psi(x-ct).$$

This solutions shows us that the solution of the wave equation consists of some waves propagating left and some waves propagating left. In this case:

- $v(x,t)$ represents the waves propagating **left**.
- $w(x,t)$ represents the waves propagating **right**.

Main point: On the boundary, $x = L$, we only want waves propagating to the right! Hence, we want to make sure that v vanishes on the boundary.

Main Point

Main point: On the boundary, $x = L$, we only want waves propagating to the right! Hence, we want to make sure that v vanishes on the boundary:

$$v(x,t) = 0, \quad x = L, \ t > 0 \tag{5}$$

which means

$$B(u) = \frac{\partial u}{\partial t} + \frac{1}{c}\frac{\partial u}{\partial x} = 0, \quad x = L, \ t > 0 \tag{6}$$

is the boundary conditions we were seeking!

The Full Problem

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0, \quad x > 0,\ t > 0 \tag{7a}$$

$$u(x,0) = U(x) \tag{7b}$$

$$\frac{\partial}{\partial t} u(x,0) = V(x) \tag{7c}$$

$$u(0,t) = 0, \tag{7d}$$

$$\frac{\partial u}{\partial t} + \frac{1}{c}\frac{\partial u}{\partial x} = 0, \quad x = L,\ t > 0 \tag{7e}$$

Finite Element Approximation

We want to be able to impose this boundary condition in our numerical scheme

Weak Formulation

v is a test function. Multiplying and integrating (1a) gives:

$$\int_\Omega u_{tt} v \, dx - \frac{1}{c^2}\int_\Omega \Delta u v \, dx = \int_\Omega u_{tt} v \, dx + \frac{1}{c^2}\left(\int_\Omega \nabla u \nabla v \, dx - \int_{\partial\Omega} \frac{\partial u}{\partial n} v \, dx\right)$$

We use the solution anzats

$$u = \sum_{i=0}^{\infty} N_i u_i(t), \quad v(x,t) = \sum_{j=0}^{\infty} N_j v_j(t), \tag{8}$$

with N_i as our element basis. Consequently

$$u_{tt} = \sum_{i=0}^{\infty} N_i \ddot{u}_i(t)$$

Weak Formulation

Inserting our solution ansatz gives the infinite dimensional system to solve

$$\sum_{i,j} \underbrace{\left(\int_\Omega N_i N_j \, dx \right)}_{=M_{i,j}} \ddot{u}_i v_j + \frac{1}{c^2} \sum_{i,j} \left(\underbrace{\int_\Omega \nabla N_i \nabla N_j \, dx}_{K_{i,j}} - \underbrace{\int_{\partial\Omega} \frac{\partial N_i}{\partial n} N_j \, dx}_{=B_{i,j}} \right) u_i vj$$

Which gives us the folowing problem

$$M\ddot{u} + Ku - Bu = 0 \qquad (9)$$

By approximating our solution u and the test function v by

$$u(x,t) \approx U(x,t) = \sum_{i=0}^{n} N_i(x) U_i(t), \tag{10}$$

$$v(x,t) \approx V(x,t) = \sum_{j=0}^{n} N_j(x) V_j(t), \tag{11}$$

we get a finite dimensional linear system of equations;

$$M\ddot{U} + KU - BU = 0. \tag{12}$$

Imposing ABC

we now take a step back in the above calculations and only consider the boundary term;

$$\frac{1}{c}\int_{\partial\Omega}\frac{\partial u}{\partial n}v\,dx = \underbrace{\frac{1}{c}\int_{\Gamma_r}\frac{\partial u}{\partial x}v\,dx}_{\text{Neumann}=g} + \frac{1}{c}\int_{\Gamma_a}\frac{\partial u}{\partial x}v\,dx$$

Where Γ_r denotes the reflecting boundary $x = 0$, and Γ_a denotes the absorbing at $x = L$.
From (6) we have the following on the absorbing boundary

$$\frac{\partial u}{\partial t} + \frac{1}{c}\frac{\partial u}{\partial x} = 0 \Rightarrow \frac{1}{c}\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial t}.$$

Therefore the boundary term becomes

$$\frac{1}{c}\int_{\Gamma_a}\frac{\partial u}{\partial x}v\,dx = -\frac{1}{c}\int_{\Gamma_a}\frac{\partial u}{\partial t}v\,dx$$

Imposing ABC

This expression

$$\frac{1}{c}\int_{\Gamma_a}\frac{\partial u}{\partial x}v\,dx = -\frac{1}{c}\int_{\Gamma_a}\frac{\partial u}{\partial t}v\,dx$$

becomes, with our solution ansatz;

$$-\int_{\Gamma_a}\frac{\partial u}{\partial t}v\,dx = \underbrace{\sum_{i,j}\left(\int_{\Gamma_a}N_iN_j\,dx\right)}_{B^a}\dot{u}_i v_i$$

Note: our boundary in the one-dimensional case is simply the end point, $x = L$.

Imposing ABC - Summary

Our final linear system of looks like this:

$$M\ddot{U} + KU - g - B^a\dot{U} = 0, \tag{13}$$

$B^a_{n,n} = \beta$ and $B^a_{i,j} = 0$ else. For the time stepping any suitable scheme (such as Newmark) can be used.

Perfectly Matched Layer

Perfectly Matched Layer (PML) works by creating an **absorbing layer** in shape of **analytical continuation** around the **region of interest**.

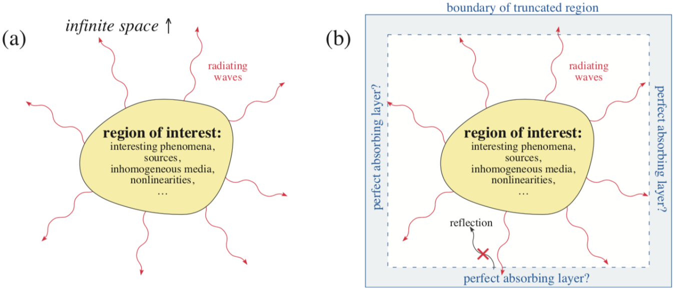


Figure 4: [**?**]

The Wave Equation

One-dimensional

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0, \quad x > 0,\ t > 0$$
$$u(x,0) = U(x)$$
$$\frac{\partial}{\partial t} u(x,0) = V(x)$$
$$u(0,t) = 0,$$

This formulation leads to solutions with the following properties:

- Reflects at the left boundary, $x = 0$
- Propagates to the right towards infinity

But...

We need to come up with a layer where waves decay sufficiently fast. Hence, we need to define a domain in which the solution decays:

$$x \mapsto z, \quad \text{for} \quad x > L.$$

The goal is now to find such a transformation such that

- The solution $u(z) \to 0$ sufficiently fast
- The solution is unharmed for $x < L$
- The whole process is computationally feasible

The Approach

Auxiliary function, $v(x, t)$, that satisfies

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial x}, \quad \frac{\partial v}{\partial t} = \frac{\partial u}{\partial x} \tag{14}$$

Then for the wave equation can be written as

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

Solution

This linear system of PDE's has the solution in infinite space

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} e^{i(kx-\omega t)},$$

with w_1 and w_2 the constant amplitudes, ω the angular frquency and k the wavenumber.

Analytical Continuation

The solution

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} e^{i(kx-\omega t)},$$

is analytic in x. Hence we can freely *analytically continue* it and evaluate in the complex plane! Then we have

$$e^{i(kx-\omega t)} = e^{ik(\mathrm{Re}x+i\mathrm{Im}x)}e^{-i\omega t} = \underbrace{e^{ik\mathrm{Re}x}e^{-i\omega t}}_{\text{Oscillating}} \underbrace{e^{-k\mathrm{Im}x}}_{\text{Decaying in } x}$$

Exponentially decaying in space for $k > 0$!

Analytical Continuation

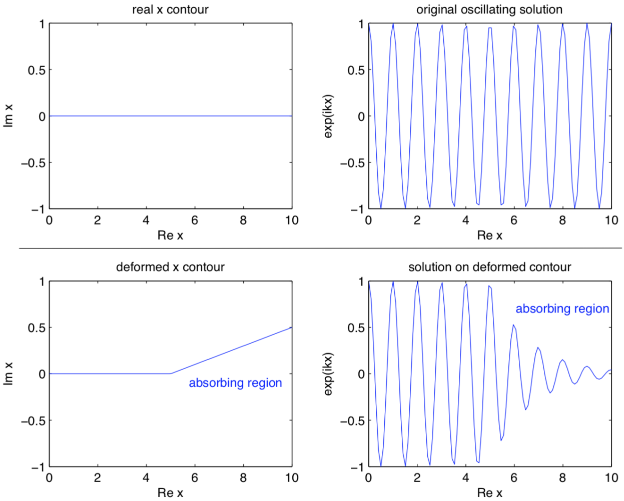


Figure 5: [?]

Pros and Cons

ABC

- Relatively Easy to derive.
- Implementation (often) comes down to well-known techniques.
- Boundary conditions can be very complicated.
- Becomes *very* complicated in higher dimensions.
- Can be unstable at inhomogeneous boundaries.

PML

- Does not reflect at the interface.
- Usually better for numerics than ABC.
- Only reflectionless for *exact* wave equation.
- Determining the complex curve can be very deficult.
- Change of coordinate can be hard to derive.
- Fails in inhomogeneous media.

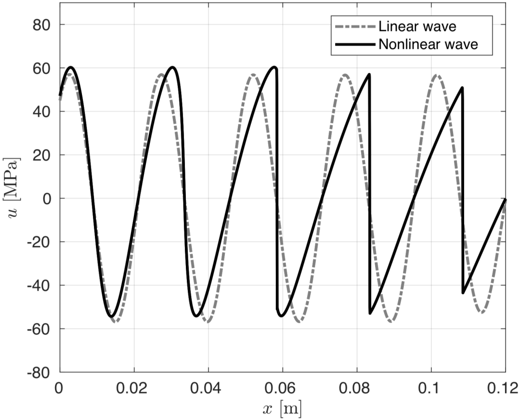
Non-Linear Absorbing Boundary Condition

Why is non-linearity a problem?

- General problems when solving numerically
- Difficult to identify in- and outgoing waves

Westervelt Equation

$$c^{-2}u_{tt} - \Delta u - \beta \Delta u_t = \gamma(u^2)_{tt} \tag{15}$$



Strategy

1. Linearise
2. Express linearised equation in terms of Pseudo-differential operators
3. Derive ABC's

But first... A (very!) Brief Introduction to Ps.d.o's

- An extension of differential operators
- Gives us a way to derive generalised inverse operators
- Makes us able to derive general properties of solutions to PDE's
  - Existence
  - Regularity
- We can derive many of these properties by looking at polynomials

Example - Helmholtz

$$P(x,D)u = (\Delta + k^2)u = 0, \quad k > 0, \quad x \in \mathbb{R}^n$$

Fourier transformation - Differentiation corresponds to multiplication!

$$\mathcal{F}(P(x,D)u) = (k^2 - \xi^2)\hat{u}(\xi) = 0$$

Inverse Fourier transformation

$$\mathcal{F}^{-1}\left[\mathcal{F}(P(x,D)u)\right] = P(x,D)u = \int (k^2 - \xi^2)\hat{u}(\xi)e^{i\xi x}\, d\xi = 0$$

Example - Helmholtz

$$P(x, D)u = \int (k^2 - \xi^2)\hat{u}(\xi)e^{i\xi x}\, d\xi = 0$$

Let's denote $p(x, \xi) = (k^2 - \xi^2)$ and call it the **symbol** of the operator. This means that our ps.d.o is defined by

$$P(x, D)u = \int p(x, \xi)\hat{u}(\xi)e^{i\xi x}\, d\xi$$

In other words; the **Pseudo-differential Operator** is the **Fourier Integral Representation** of the differential operator.

Composition of Ps.d.o's

The composition (product) of two ps.d.o's is a new ps.d.o;

$$P_1 \circ P_2 = P \tag{16}$$

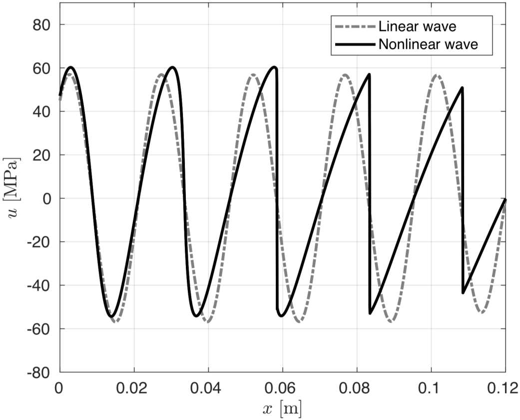
with symbol satisfying

$$p(x,\xi) \sim \sum_{\alpha \geq 0} \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha p_1(x,\xi) D_x^\alpha p_2(x,\xi) \tag{17}$$

For our purposes, we can switch the " $\sim$ " with " $=$ ".

Back to Westervelt Equation!

$$c^{-2}u_{tt} - \Delta u - \beta \Delta u_t = \gamma (u^2)_{tt} \tag{18}$$



Linearisation

$$c^{-2}u_{tt} - \Delta u - \beta\Delta u_t = \gamma(u^2)_{tt} \tag{19}$$

We linearise by inserting a reference solution, $u^{(0)}(x)$, and add a small perturbation $\epsilon w(x,t)$;

$$c^{-2}\left(u^{(0)} + \epsilon w(x,t)\right)_{tt} - \Delta\left(u^{(0)} + \epsilon w(x,t)\right)$$
$$-\beta\Delta\left(u^{(0)} + \epsilon w(x,t)\right)_t = \gamma(\left(u^{(0)} + \epsilon w(x,t)\right)^2)_{tt}.$$

Doing some reductions and ignore ϵ^2 terms...

Linearisation

Finally, by replacing w by u, we get

$$\left(c^{-2} - 2\gamma u^{(0)}\right) u - \Delta u - \beta \Delta u_t = 2\gamma u_t^{(0)} u_t$$
$$\Rightarrow \nu^2 u_{tt} - u_{xx} - \beta u_{txx} - 2\gamma u_t^{(0)} u_t = 0$$

with $\nu^2 = \nu^2(u^{(0)})$, where $\nu^2(v) = c^{-2} - 2\gamma v$.
The operator form is given by

$$D_1 u = 0, \quad D_1 = \nu^2 \partial_t^2 - \partial_x^2 - \beta \partial_{txx} - 2\gamma u_t^{(0)} \partial_t \tag{20}$$

- $c^{-2} - 2\gamma u = \nu^2(u) > 0$,
- $\nu \in C^\infty$

Pseudo-Differential Representation

We now want to identify **in- and outgoing waves** as we did in the linear wave equation. But our problem is much more **complicated**!

Nirenberg Factorisation

We can factorise D_1;

$$D_1 = -(\partial_x - A)(\partial_x - B) + R \tag{21}$$

where

$$A = A(x, t, D_t), \quad B = B(x, t, D_t)$$

are ps.d.o's with symbols

$$a(x, t, \tau) = \sum_{j=0}^{k} a_j(x)\tau^j, \quad b(x, t, \tau) = \sum_{j=0}^{k} b_j(x)\tau^j$$

and $D_t = -i\partial_t$. Furthermore, R is a smoothing ps.d.o; which means that we can "neglect" it.

$$\nu^2\partial_t^2 - \partial_x^2 - \beta\partial_{txx} - 2\gamma u_t^{(0)}\partial_t = -(\partial_x - A)(\partial_x - B) + R$$

Computing the Symbols

How does A and B look like? - or more specifically: how does their symbols, a and b, look like?

Since we want to compute **explicit boundary conditions** we need to compute the symbols to gain anything from the ps.d.o representation!

Computing the Symbols

We start by developing the factorisation

$$D_1 = -(\partial_x - A)(\partial_x - B) + R = -\partial_x^2 + (A + B)\partial_x + B_x - AB + R$$

Which at the symbolic level is

$$d_1(x, t, \tau) = -\partial_x^2 + (a + b)\partial_x + b_x - ab + R$$

... and the symbolic representation of the **non**-ps.d.o. representation is given by

$$\nu^2(i\tau)^2 - \partial_x^2 - \beta(i\tau)\partial_{xx} - 2\gamma u_t^{(0)}(i\tau)$$

obtained by Fourier transformation in time.

Combing the two symbol representations:

$$\nu^2(i\tau)^2 - \partial_x^2 - \beta(i\tau)\partial_{xx} - 2\gamma u_t^{(0)}(i\tau) = -\partial_x^2 + (a+b)\partial_x + b_x - ab + R$$
$$\Rightarrow \nu^2(i\tau)^2 - \beta(i\tau)\partial_{xx} - 2\gamma u_t^{(0)}(i\tau) = (a+b)\partial_x + b_x - ab + R$$

We can use this equality to compute a **and** b**!**

But first... What is *ab*?

We use the theorem from earlier to represent the symbol of the composition of of two ps.d.o's:

$$ab(x,t,\tau) \sim \sum_{k,l,n\geq 0} \frac{(-i)^n}{n!} \partial_\tau^n a_{1-k}(x,t,\tau) \partial_\tau^n b_{1-l}(x,t,\tau)$$
$$= \sum_{j\geq 0,\ k+l+n=j} \frac{(-i)^n}{n!} \partial_\tau^n a_{1-k}(x,t,\tau) \partial_\tau^n b_{1-l}(x,t,\tau)$$

This is a polynomial of order $\mathcal{O}(\tau^{2-j})$.

... which means

$$(a+b)\partial_x + b_x - ab + R = \sum_{j\geq 0}(a_{1-j} + b_{1-j})\partial_x + \sum_{j\geq 0}\partial_x b_{1-j})$$
$$- \sum_{j\geq 0,\ k+l+n=j} \frac{(-i)^n}{n!}\partial_\tau^n a_{1-k}\partial_\tau^n b_{1-l}$$

$$\nu^2(i\tau)^2 - \beta(i\tau)\partial_{xx} - 2\gamma u_t^{(0)}(i\tau) = \sum_{j\geq 0}(a_{1-j} + b_{1-j})\partial_x + \sum_{j\geq 0}\partial_x b_{1-j} - \sum_{j\geq 0,\ k+l+n=j} \underbrace{\frac{(-i)^n}{n!}\partial_\tau^n a_{1-k}\partial_\tau^n b_{1-l}}_{\mathcal{O}(\tau^{2-j})}$$

To compute a_i and b_i we **match coeffcients of** τ^j. The more coefficients we take the higer accuracy of ABC! - but also higher complexity.

$$\mathcal{O}(\tau^2), j = 0: \quad \begin{cases} a_1 + b_1 = 0 \\ \nu^2(i\tau)^2 = -a_1 b_1 \end{cases}$$

$$\mathcal{O}(\tau), j = 1: \quad \begin{cases} a_0 + b_0 = 0 \\ \beta(i\tau)\partial_x^2 + 2\gamma u_t^{(0)}(i\tau) = a_1 b_0 + a_0 b_1 - i a_{1\tau} b_{1t} - b_{1x} \end{cases}$$

$$\mathcal{O}(\tau^0), j = 2: \quad \begin{cases} a_{-1} + b_{-1} = 0 \\ -a_1 b_{-1} - a_0 b_0 - a_{-1} b_1 + i(a_{1\tau} b_{0t} + a_{0\tau} b_{1t}) \\ -\frac{i^2}{2} a_{1\tau\tau} b_{1tt} + b_{0x} = 0 \end{cases}$$

$$a_1 = -\nu(i\tau), \quad a_0 = \frac{1}{2\nu}\left(G\nu + 2\gamma u_t^{(0)}\right), \quad a_{-1} = \frac{\gamma\mu}{2\nu(i\tau)}$$

$$b_1 = \nu(i\tau), \quad b_0 = -\frac{1}{2\nu}\left(G\nu + 2\gamma u_t^{(0)}\right), \quad b_{-1} = -\frac{\gamma\mu}{2\nu(i\tau)}$$

where

$$\mu = G\left(\frac{1}{2\nu}\left(G\nu + 2\gamma u_t^{(0)}\right)\right) - \left(\frac{1}{2\nu}\left(G\nu + 2\gamma u_t^{(0)}\right)\right)^2$$

and $G = (\partial_x + \nu\partial_t)$.
Note: $a_i = -b_i$

And therefore

$$D_1 = -(\partial_x - A)(\partial_x - B) + R = \nu^2 \partial_t^2 - \partial_x^2 - \beta \partial_{txx} - 2\gamma u_t^{(0)} \partial_t$$

has the symbol

$$\begin{aligned} -(\partial_x - a)(\partial_x - b) + R &= -(\partial_x - a)(\partial_x + a) + R \\ &= -\left(\partial_x - \sum_{j=0}^{2} a_{1-j}\right)\left(\partial_x + \sum_{j=0}^{2} a_{1-j}\right) + R \end{aligned}$$

with

$$a_1 = -\nu(i\tau), \quad a_0 = \frac{1}{2\nu}\left(G(\nu) + 2\gamma u_t^{(0)}\right), \quad a_{-1} = \frac{\gamma\mu}{2\nu(i\tau)}$$

Summary

What have we done so far?

- Linearised our PDE

$$D_1 = \nu^2 \partial_t^2 - \partial_x^2 - \beta \partial_{txx} - 2\gamma u_t^{(0)} \partial_t$$

- Formulated the PDE in terms of ps.d.o's

$$D_1 = -(\partial_x - A)(\partial_x - B) + R,$$

- Computed the symbols of A and B and realised

$$D_1 = -(\partial_x - A)(\partial_x + A) + R,$$

Absorbing Boundary Conditions

We have now done the hard work!

The ps.d.o. representation,

$$D_1 = -(\partial_x - A)(\partial_x + A) + R,$$

makes it possible for us to identify in- and outgoing waves. According to [**?**], the operator

$$\partial_x + a(x, t, D_t)$$

represents left-propagating waves!

Absorbing Boundary Conditions

... Which means

$$\partial_x + a(x, t, D_t) = 0$$

is the operator we need at the boundary! This gives us

$$\left(\partial_x + \sum_{j=0}^{k} a_{1-j}(x, t, D_t) \right) u \Big|_{x=L} = 0$$

k denotes the order of our ABC.

0th order ABC

$k = 0$. We use only a_1:

$$\left(\partial_x + \sum_{j=0}^{0} a_{1-j}(x,t,D_t)\right) u\Big|_{x=L} = \left(\partial_x + \nu\partial_t\right) u\big|_{x=0} = L$$

This almost the same as for the linear wave equation!

Note: Remember, $a_1(x,t,\tau) = \nu(i\tau)$ corresponds to $a_1(x,t,D_t) = -\nu\partial_t$.

1st order ABC

$k = 1$. We use a_1 and a_0:

$$\left(\partial_x + \sum_{j=0}^{1} a_{1-j}(x,t,D_t) \right) u \Bigg|_{x=L}$$

$$= \left(u_x + \nu u_t - \frac{1}{2\nu} \left((\nu_x + \nu \nu_t) u + 2\gamma u_t^{(0)} u \right) \right) \Bigg|_{x=L} = 0$$